Fourier Analysis (formulae and exercises)

enric.meinhardt@cmla.ens-cachan.fr

0. Prerequisites

0.1. Differential and integral calculus of one and several variables

0.2. Complex numbers: $e^{i\theta} = \cos\theta + i\sin\theta$, etc

0.3. Taylor series: $f(x) = c_0 + c_1 x + c_2 + x^2 + \cdots \quad \Longrightarrow \quad c_n = \frac{f^{(n)}(0)}{n!}$

0.4. Linear algebra, orthonormal bases: $\mathbf{v} = a_1\mathbf{e}_1 + \cdots + a_n\mathbf{e}_n \quad \Longrightarrow \quad a_i = \langle \mathbf{v}, \mathbf{e}_i \rangle$

0.5. Given the graph of a function, draw the graphs of its derivative and integral.

1. Fourier transform of functions $f : \mathbf{R} \to \mathbf{C}$

1.1 Definitions. The Fourier transform of an integrable function $f : \mathbf{R} \to \mathbf{C}$ is the function $\hat{f} : \mathbf{R} \to \mathbf{C}$ defined by

$$\widehat{f}(y) := \int f(x) e^{-ixy} \mathrm{d}x$$

The inverse Fourier transform of an integrable function $g : \mathbf{R} \to \mathbf{C}$ is the function

$$\check{g}(x) := \frac{1}{2\pi} \int g(y) e^{ixy} \mathrm{d}y$$

The Fourier inversion theorem says that (for a certain class of functions) these two operations are inverses of each other. Thus, we have a representation of f as an (infinite) linear combination of sinusoidal functions, where the coefficients of the linear combination are given by $\widehat{f}(y)$:

$$f(x) = \frac{1}{2\pi} \int \widehat{f}(y) e^{ixy} \mathrm{d}y$$

With some care, these definitions are extended to all square-integrable functions (not necessarily integrable).

1.2. Energy conservation (Plancherel's theorem)

$$\int \left| f(x) \right|^2 \mathrm{d}x = \frac{1}{2\pi} \int \left| \widehat{f}(y) \right|^2 \mathrm{d}y$$

1.3. Convolution theorems

$$\widehat{f * g} = \widehat{f} \cdot \widehat{g} \qquad\qquad \widehat{fg} = \widehat{f} * \widehat{g}$$

Where the convolution of two functions is defined by

$$(f * g)(y) = \int f(x) g(y - x) \mathrm{d}x$$

1.4. General properties

f	$\widehat{f}$
real even	real even
real odd	imaginary odd
real	complex hermitian $\overline{\hat{f}(y)} = \hat{f}(-y)$
$\lambda f + \mu g$	$\lambda\widehat{f} + \mu\widehat{g}$
$f(x/a)$	$\lvert a\rvert\widehat{f}(ay)$
$f(x-a)$	$e^{-iay}\widehat{f}(y)$
$f'(x)$	$-iy\widehat{f}(y)$

1.5. Examples of transforms

$f(x)$	$\widehat{f}(y)$
$\chi_{[-\frac{1}{2a},\frac{1}{2a}]}(x)$	$\frac{1}{\lvert a\rvert}\mathrm{sinc}\left(\frac{y}{2\pi a}\right)$
$e^{-ax}\mathrm{H}(x)$	$\frac{1}{a+iy}$
$e^{-a\lvert x\rvert}$	$\frac{2a}{a^2+y^2}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{y^2}{4a}}$
$\mathrm{sech}\,(ax)$	$\frac{\pi}{a}\mathrm{sech}\left(\frac{\pi}{2a}y\right)$

$f(x)$	$\widehat{f}(y)$ (in the sense of distributions)
1	$2\pi\delta(y)$
$\delta(x)$	1
e^{iax}	$2\pi\delta(y-a)$
$\cos(ax)$	$\pi\delta(y-a)+\pi\delta(y+a)$
$\sin(ax)$	$-i\pi\delta(y-a)+i\pi\delta(y+a)$
x^n	$2\pi i^n\delta^{(n)}(y)$
$\sum_{n\in\mathbf{Z}}\delta(x-na)$	$\frac{2\pi}{a}\sum_{k\in\mathbf{Z}}\delta\left(y-\frac{2\pi k}{a}\right)$

2. Fourier series of functions $f : \mathbf{T} \to \mathbf{C}$

Let $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ be the periodization of the interval $[0, 2\pi]$. The functions $f : \mathbf{T} \to \mathbf{C}$ are identified with the the 2π-periodic functions on $\mathbf{R}$. They can be expressed as Fourier series, which is a linear combination of sinusoidal functions of integer frequencies:

$$f(\theta) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)\exp^{in\theta} \tag{1}$$

the coefficients $\widehat{f}(n)$ are

$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\exp^{-in\theta}\,\mathrm{d}\theta \tag{2}$$

(this formula follows by multiplying formula (1) with the function $e^{-in\theta}$ and integrating with respect to θ.)

Fourier series have analogous properties to Fourier integrals (see 1.4. above). Only the property for $f(x/a)$ has no direct equivalent, because it does not preserve the periodicity of the function.

3. Discrete Fourier transform of functions $f : \mathbf{Z}_N \to \mathbf{C}$

The "functions" $f : \mathbf{Z}_N \to \mathbf{C}$ are the vectors of $\mathbf{C}^N$. The set of vectors

$$\mathbf{e}_n = \left(e^{\frac{2\pi i k n}{N}} \right)_{k=0,\dots,N-1}$$

for $n = 0, \dots, N-1$, is an orthogonal basis of $\mathbf{C}^N$, indeed $\mathbf{e}_p \cdot \mathbf{e}_q = N\delta_{pq}$.

The discrete Fourier transform (DFT) is the expression of $\mathbf{C}^N$ vectors in this basis. Thus, a vector $\mathbf{v} = (v_0, \dots, v_{N-1})$, can be expressed as

$$v_k = \sum_{n=0}^{N-1} \widehat{v}_n e^{\frac{2\pi i k n}{N}}$$

and the coefficients $\widehat{v}_n$ are recovered by computing $\mathbf{v} \cdot \mathbf{e}_n$:

$$\widehat{v}_n = \frac{1}{N} \sum_{k=0}^{N-1} v_k e^{-\frac{2\pi i k n}{N}}$$

The discrete Fourier transform has analogous properties to the other transforms. Here all the relationships are trivial to check using linear algebra (there are not convergence problems as in the previous cases).

4. Sampling

If $f : \mathbf{T} \to \mathbf{C}$ is of the form $f(\theta) = a_0 + a_1 e^{i\theta} + \ldots + a_{N-1} e^{i(N-1)\theta}$ we say that it is a band-limited function, or a trigonometric polynomial. The coefficients a_k can be obtained from $f(\theta)$ by computing its Fourier series (which is a finite sum).

Sampling theory provides another way to compute these coefficients. First, we evaluate the function f at N equally spaced points $\mathbf{v} := \big(f(2\pi k/N)\big)_{k=0,\dots,N-1}$. Then, the vector of coefficients $\mathbf{a} = (a_0, \dots, a_{N-1})$ is the DFT of the vector $\mathbf{v}$.

There are similar relationships between the other transforms described above, describing how each transform commutes with sampling and interpolation operators.

5. Aliasing

Continuing with the sampling example above, suppose that $N = 2P + 1$ is odd, and notice that the two trigonometric polynomials

$$f(\theta) = a_0 + a_1 e^{i\theta} + \ldots + a_{N-1} e^{i(N-1)\theta}$$

and

$$\tilde{f}(\theta) = a_0 + a_1 e^{i\theta} + \cdots + a_P e^{iP\theta} + a_{P+1} e^{-iP\theta} + a_{P+2} e^{-i(P-1)\theta} \cdots + a_{P+P} e^{-2i\theta} + a_{P+P+1} e^{-i\theta}$$

take exactly the same values at the points $\theta_k = \frac{2\pi k}{N}$ for $k = 0, \dots, N-1$. This happens because they have the same frequencies modulo N and the functions $e^{ik\theta}$ are 2π–periodic. The second choice is much more natural (since it is the only way to obtain real-valued signals!), and it is typically written as

$$f(\theta) = \sum_{k=-\lfloor N/2 \rfloor}^{\lfloor N/2 \rfloor - 1} a_k e^{ik\theta}$$

where the indices of the coefficients a_k are to be understood modulo N.

This is the simplest case of aliasing: different functions having the same samples. In the general case, we sample a function $f(\theta)=\sum_{n=0}^{N-1}a_ne^{in\theta}$ at M points $\theta_k=\frac{2\pi k}{M}$. The case $M=N$ is the perfect sampling rate (corresponding to the Shannon-Nyquist condition), and the coefficients a_n are the DFT of the samples.

The case $M>N$ is called *oversampling*, *zero-padding* or *zoom-in*, depending on the context. In that case we have

$$f(\theta_k)=\sum_{n=0}^{N-1}a_ne^{\frac{2\pi ikn}{M}}+\sum_{n=N}^{M-1}0\cdot e^{\frac{2\pi ikn}{M}}=\sum_{n=0}^{M-1}\mathrm{ZP}(a)_ne^{\frac{2\pi ikn}{M}}$$

where $\mathrm{ZP}(a_0,a_1,\ldots,a_{N-1})=(a_0,\ldots,a_{N-1},0,\ldots,0)$ is the zero-padding of the vector a to length M.

The case $M<N$ is called *subsampling*, *aliasing*, *decimation* or *zoom-out*, depending on the context. In that case we have

$$f(\theta_k)=\sum_{n=0}^{N-1}a_ne^{\frac{2\pi ik(n\%M)}{M}}=\sum_{m=0}^{M-1}\left(\sum_{n\%M=m}a_n\right)e^{\frac{2\pi ikm}{M}}=\sum_{m=0}^{M-1}\mathrm{AL}(a)_me^{\frac{2\pi ikm}{M}}$$

where $\mathrm{AL}(a)$ is the vector a *folded* to length M by summing the positions that are equal modulo M.

6. Fourier transform in several dimensions

If $f:\mathbf{R}^N\to\mathbf{C}$ is an integrable function, its Fourier transform is defined as:

$$\widehat{f}(\mathbf{y})=\int_{\mathbf{R}^n}f(\mathbf{x})e^{-i\mathbf{x}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{x}$$

and the inverse transform is

$$f(\mathbf{x})=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\widehat{f}(\mathbf{y})e^{i\mathbf{x}\cdot\mathbf{y}}\,\mathrm{d}\mathbf{y}$$

The properties are directly derived from those of the 1-dimensional Fourier transform by separability.

7. (Optional) Generalization: Pontryagin duality

The transforms described above are particular cases of a general construction called Pontryagin duality, which works on locally compact abelian groups G. On such a group, there is a natural measure called the *Haar measure* that allows to compute integrals of functions $f:G\to\mathbf{C}$. The *dual* of a group G is the set of morphisms $\mu:G\to\mathbf{C}^*$ (here C^* is the complex unit circle). The dual G^* is itself a group under the point-wise product of functions. Moreover, it is locally compact and abelian, and it has its own Haar integral. Finally, the Fourier transform of an integrable function $f:G\to\mathbf{C}$ is a function $\hat{f}:G^*\to\mathbf{C}$ defined as

$$\hat{f}(y)=\int_G f(x)\overline{y(x)}\mathrm{d}x \tag{3}$$

And, likewise, given an integrable function $g:G^*\to\mathbf{C}$, its inverse transform is

$$\check{f}(x)=\int_{G^*}f(y)y(x)\mathrm{d}y \tag{4}$$

And the Pontryagin duality theorem states that these two operations are inverses of each other. Notice that since $y(x)$ is a unit complex number, it is typically written as e^{iyx}.

On the table below we find the typical transforms as particular cases

	spatial domain	frequency domain	analysis	synthesis
general case	G	G^*	$\hat{f}(y) = \int_G f(x)e^{-iyx}dx$	$\check{f}(x) = \int_{G^*} f(y)e^{iyx}dy$
Fourier series	$\mathbf{T}$	$\mathbf{Z}$	$\hat{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}d\theta$	$\check{f}(\theta) = \sum_{n\in\mathbf{Z}} f_n e^{inx}$
Fourier transform	$\mathbf{R}$	$\mathbf{R}$	$\hat{f}(y) = \int_{\mathbf{R}} f(x)e^{-iyx}dx$	$\check{f}(x) = \frac{1}{2\pi}\int_{\mathbf{R}} f(y)e^{iyx}dy$
DFT	$\mathbf{Z}_N$	$\mathbf{Z}_N$	$\hat{f}_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-ikn}$	$\check{f}_n = \sum_{n=0}^{N-1} f_k e^{ikn}$
DTFT	$\mathbf{Z}$	$\mathbf{T}$	$\hat{f}(\theta) = \sum_{n\in\mathbf{Z}} f_n e^{-inx}$	$\check{f}_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{in\theta}d\theta$

Notice that the placement of many multiplicative factors in this table seems arbitrary. Indeed the Haar measure is unique up to a multiplicative constant, and this leads to different conventions for the factors.

8. (Optional) Generalization: Laplace-Beltrami spectrum

Another generalization of Fourier series and integrals is found in differential geometry. Given a manifold Ω (for example, a subset of the plane, or an arbitrary curved surface), a standard geometric construction is the Laplace-Beltrami operator $\Delta_\Omega : C^\infty(\Omega) \to C^\infty(\Omega)$. This is a positive-definite linear operator, and when Ω is compact, it has a numerable set of eigenfunctions f_k, satisfying $\Delta_\Omega f_k = \lambda_k f_k$, with $0 < \lambda_1 \leq \lambda_2 \leq \ldots$. Under mild regularity conditions, these functions form an orthogonal basis of $L^2(\Omega)$.

The functions f_k are called the *harmonics*, and the numbers λ_k are called the *fundamental frequencies*. In the case of a solid body Ω, they correspond to the modes of vibration of the object. For example, if Ω is the skin of a drum, the functions f_k describe the shapes in which the skin can vibrate, and the numbers λ_k determine the frequency at which they vibrate. Any vibration pattern can be expressed as a linear combination of functions f_k.

Notice that when Ω is one-dimensional, the Laplace-Beltrami operator is minus the second derivative, and the solutions of $-f'' = \lambda^2 f$ are the functions of the form $f(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda y)$. On the table below we find some particular cases

Ω	eigenfunctions of Δ_Ω
$[0,\pi]$	$\sin(kx) \quad k = 1, 2, \ldots$
$[0,\pi]^2$	$\sin(px)\sin(qy) \quad p, q = 1, 2, \ldots$
unit disk	Bessel functions
sphere	spherical harmonics
drum skin	harmonics of the drum

9. Application : solving linear PDE

Given a function $f : \mathbf{R}^2 \to \mathbf{R}$, we want to find a function u satisfying the following PDE:

$$u - \alpha^2 \Delta u = f$$

This is a linear PDE with constant coefficients. We can solve it by applying the Fourier transform on each side of the equation to obtain an equivalent relation:

$$\hat{u} + \alpha^2(\xi^2 + \eta^2)\hat{u} = \hat{f}$$

And solving for $\hat{u}$

$$\hat{u} = \frac{1}{1 + \alpha^2(\xi^2 + \eta^2)} \cdot \hat{f}$$

we find the Fourier transform of the solution. Thus the solution is the convolution of the datum f with a positive kernel of type Laplace.

More generally, a linear PDE has the form

$$\mathrm{P}\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right) u = f$$

where P is a polynomial in n variables. Applying the Fourier transform on both sides, we obtain

$$\mathrm{P}\left(i\xi_1, \ldots, i\xi_n\right) \hat{u} = \hat{f}$$

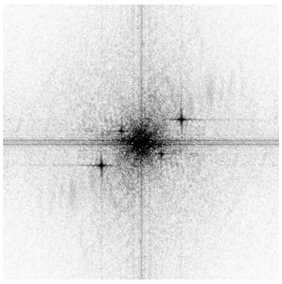
Which gives the solution immediately.

10. Application : image processing

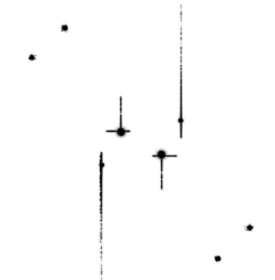
The Fourier transform in dimension 2 is an important tool in image processing.

A simple application is the removal of periodic noise in images. The frequencies of periodic noise on an image I appear as local maxima in the image $|\widehat{\mathrm{I}}|$, and can be removed manually (setting them to zero using an image editor).

I



$\log\left(1 + |\widehat{\mathrm{I}}|\right)$



mask for $\widehat{\mathrm{I}}$

reconstruction

11. Bibliography

11.0. T. W. Körner : *Fourier analysis*
(learn all the history in well-written, very short lessons)

11.1. R. Bracewell : *The Fourier Transform and its Applications*
(lean to compute Fourier transforms visually and graphically)

11.2. C. Gasquet, P. Witomski : *Analyse de Fourier et applications*
(the complete mathematical theory in dimension 1, from the elementary to the fancier)

11.3. R. C. Gonzalez, R. E. Woods : *Digital Image Processing*
(applications to image processing, with practical emphasis on programming)

12. Exercices (stars denote relative difficulty)

E0. Is it possible to sample pure sinusoidal wave of frequency 10.000Hz (a very high pitched sound) so that you hear a pure sinusoidal wave of frequency 440Hz (the middle A). If that is the case, what is the necessary sampling rate?

E1. Is the definition of the Fourier transform (section 1.1) correct? Do all integrable functions (i.e., those that $\int_{\mathbf{R}} |f| < \infty$) have a well-defined Fourier transform? Is $\hat{f}(\xi)$ a bounded function? Is it continuous?

E2. Check the general properties stated on section 1.4.

E3. Check the validity of the convolution theorem (recall the definition of convolution $(f * g)(x) := \int_{\mathbf{R}} f(x-y)g(y)\mathrm{d}y$). What hypotheses are needed on f and g to assure that the statement makes sense?

E4. Check the first three Fourier transforms on the table 1.5.

E5. Discuss the following reasoning. Let f be an integrable function. By the inversion theorem, the function f it the Fourier transform of $\check{f}$. Thus, f is continuous (by exercise E1).

***E6.** The goal of this exercice is to compute that the Fourier transform of a gaussian function is another gaussian function. Let $a > 0$ and $\psi(x) = e^{-ax^2}$.

(a) Prove that $\psi(x)$ is integrable.
(b) Compute $\hat{\psi}(0)$. (This is a classical result that you *must* know)
(c) Assume that $\hat{\psi}$ is derivable. Prove that $\widehat{\psi}\,' = -i\hat{g}$, where g is the function $x \mapsto x\psi(x)$.
(d) Prove that $\widehat{\psi'}(\xi) = i\xi\hat{\psi}(\xi)$.
(e) Combine the previous two results to obtain a differential equation for ψ, and solve it. Write an explicit formula for $\hat{\psi}(\xi)$.

E7. Check that the vectors $\mathbf{e}_n$ of section 3 are orthogonal. *(Hint: there is a geometric interpretation.)*

E8. Prove the sampling result stated on section 4 (relating the DFT to Fourier series).

E9. A *left coordinate shift* of k positions is the map $s_k : \mathbf{R}^{\mathrm{N}} \to \mathbf{R}^{\mathrm{N}}$ defined by

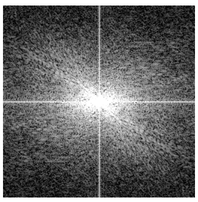
$$s_k(x_0, x_1, \ldots, x_{\mathrm{N}-1}) := (x_k, x_{k+1}, \ldots, x_{k+\mathrm{N}-1})$$

where the indices are to be interpreted modulo N. This definition makes sense only when $k \in \mathbf{Z}$. How would you define a coordinate shift for $k \in \mathbf{R}$? *(Hint: look at the various properties of the DFT).*

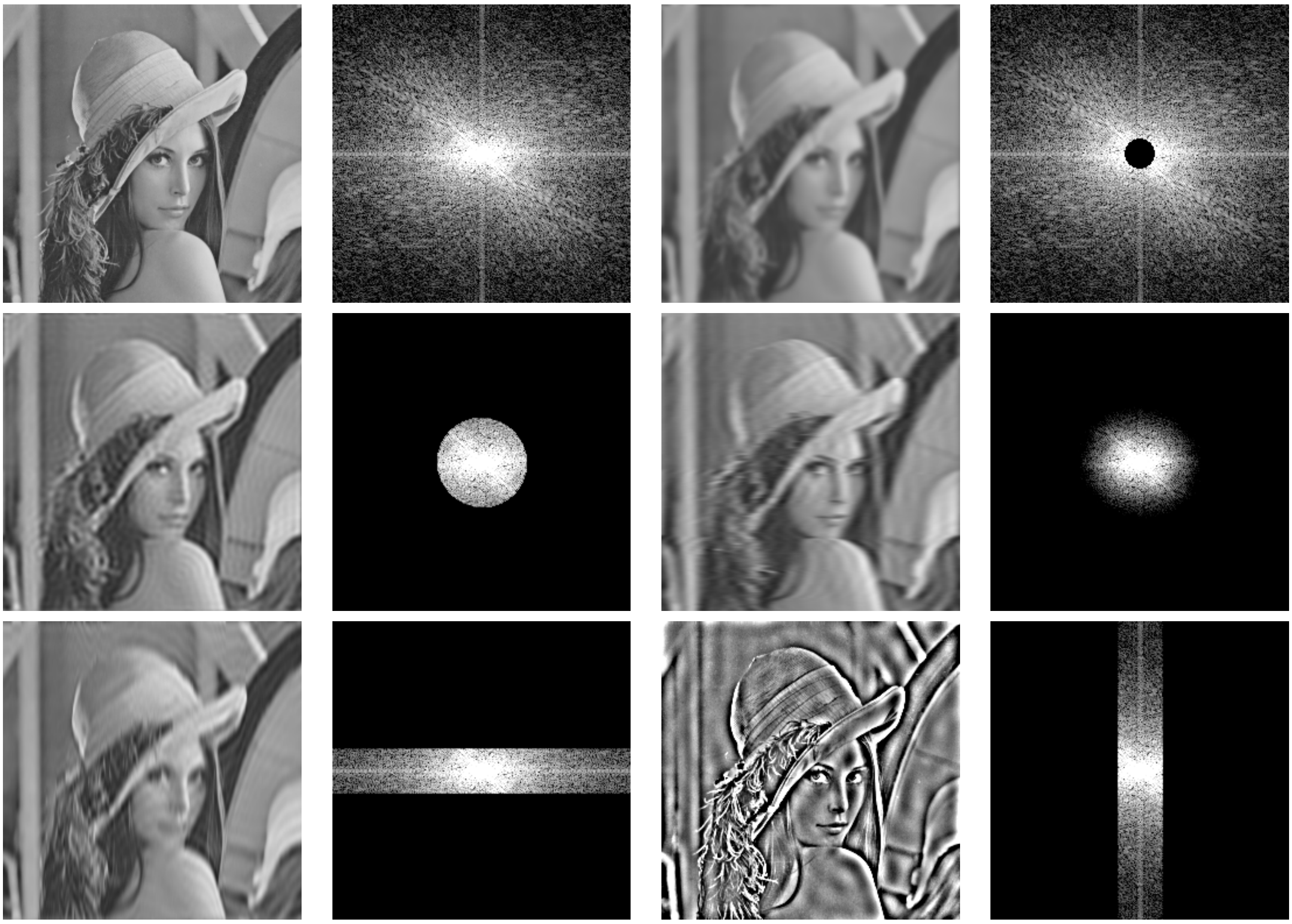
****E10.** The previous exercice gives a discrete implementation of a shift $f(x-a)$ for $a \in \mathbf{R}$. Given $a > 1$, how would you define a *zoom-in* $f(ax)$ and a *zoom-out* $f(x/a)$?

E11. A *digital image* is an array of $\mathrm{W} \times \mathrm{H}$ real numbers. The indexes of the array are called *pixels* and the value of each pixel is called its *gray level.* How would you define the Fourier transform of an image? How would you display it?

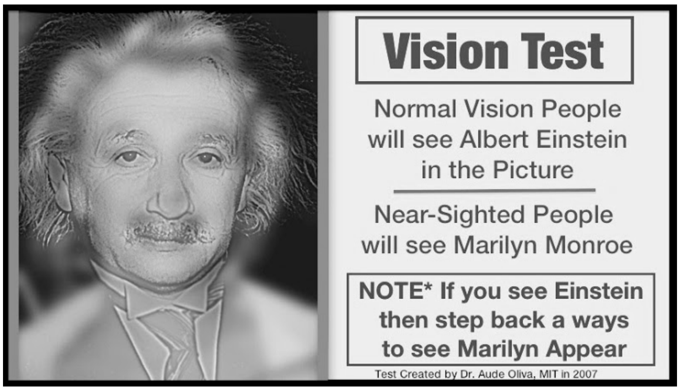
E12. The following figure shows an image and its Fourier transform. What is the value of the central pixel? Why do you see a cross around it? *(Hint: how would the original image look after a coordinate shift?)*

***E13**. Below you see six images and their six Fourier transforms (not necessarily in the same order). Look *very attentively* at the images. Which image corresponds to each spectrum?



*****E14**. Explain the following optical illusion and formalize it in terms of Fourier analysis.



****E15**. Give a closed form expression for the sum $g(x) = \sum_{n \geq 1} \frac{1}{n^2 + x^2}$. Check that $\lim_{x \to 0} = \frac{\pi^2}{6}$.
Hint: compute the Parseval identity for the Fourier series of the function $f(\theta) = e^{\beta\theta}$.