Functional Analysis

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(collection of some definitions and results)

0. Prerequsites

0.1. Differential and integral calculus of one and several variables

0.2. Linear algebra: vector spaces, linear maps, inner products, dual space

0.3. Measure theory: null sets, equality "almost everywhere", integrable functions

0.4. General topology: compactness, continuity, density

0.5. *All norms in finite dimension are equivalent*

1. Basic integration results

1.1. **Definition.**
A function $f : \Omega \to \mathbf{R}$ is *integrable* if $\int_\Omega |f| < \infty$. The set of all integrable functions on Ω is denoted $L^1(\Omega)$. Two functions of $L^1(\Omega)$ that coincide almost everywhere are considered as the same function. Using the norm $\|f\|_{L^1(\Omega)} := \int_\Omega |f|$, this set is a normed vector space (and it turns out to be complete).

1.2. **Theorem (monotone convergence, Beppo Levi)**
Let f_n an increasing sequence of positive L^1 functions such that $\sup_n \int f_n < \infty$. Then $f_n(x)$ converges almost everywhere to a finite limit denoted $f(x)$. Moreover, $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1} \to 0$.

1.3. **Theorem (dominated convergence, Lebesgue)**
Let f_n a sequence of L^1 functions converging pointwise a.e. to a function $f(x)$. Assume there exist a function $h \in L^1(\Omega)$ such that $|f_n(x)| \leq h(x)$ almost everywhere. Then $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1} \to 0$.

1.4. **Theorem (density)**
The set $C_c(\Omega)$ of compactly supported continuous functions is dense in $L^1(\Omega)$.

1.5. **Theorem (Tonelli)**
If $f : \Omega_1 \times \Omega_2 \to \mathbf{R}^+ \cup \{+\infty\}$ then

$$\iint_{\Omega_1 \times \Omega_2} f(x,y) \mathrm{d}x\mathrm{d}y = \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) \mathrm{d}y \right) \mathrm{d}x = \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y) \mathrm{d}x \right) \mathrm{d}y.$$

in particular, if one of these three integrals is $+\infty$, then the other two also are also. (In words: the sum of positive numbers, be it finite or infinite, does not depend on the order in which they are summed.)

1.6. **Theorem (Fubini)**
If $f \in L^1(\Omega_1 \times \Omega_2)$ then the three members in the formula above are well-defined and they are equal. In particular, the fact that the second member is well defined implies that
- The function $y \mapsto f(x,y)$ belongs to $L^1(\Omega_2)$ for a.e. x
- The function $x \mapsto \int_{\Omega_2} f(x,y)\mathrm{d}y$ thus defined belongs to $L^1(\Omega_1)$.

2. L^p spaces

2.1. **Definition.** Let us define the following numbers (which may be infinite)

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p\right)^{1/p} \qquad \text{for } 1 \le p < \infty$$

$$\|f\|_{L^\infty(\Omega)} := \inf\left\{M \ ; \ |f(x)| \le M \text{ a.e.on } \Omega\right\}$$

The space $L^p(\Omega)$ for $1 \le p \le \infty$ is the set of functions such that $\|f\|_{L^p(\Omega)} < \infty$.
To each $p \in [1,\infty]$ we associate its *conjugate exponent* p' such that $\frac{1}{p} + \frac{1}{p'} = 1$.

2.2. **Hölder inequality.** Let $p \in [1,\infty]$, $f \in L^p$ and $g \in L^{p'}$. Then $fg \in L^1$ and

$$\int |fg| \le \|f\|_{L^p} \, \|g\|_{L^{p'}}$$

2.3. **Minkowski's inequality.** Let $p \in [1,\infty]$ and $f, g \in L^p$. Then

$$\|f+g\|_{L^p} \ \le \ \|f\|_{L^p} \ + \ \|g\|_{L^p}$$

2.4. **Interpolation inequality.** Let $1 \le s \le r \le t \le \infty$ and $f \in L^s \cap L^t$. Then $f \in L^r$ and

$$\|f\|_{L^r} \ \le \ \|f\|_{L^s}^{\theta} \, \|g\|_{L^t}^{1-\theta}$$

where $\theta = \frac{s(t-r)}{r(t-s)}$.

2.6. **Theorem (Fischer-Riesz).** L^p is a complete normed vector space for $p \in [1,\infty]$.

2.7. **Riesz representation theorem.** Let $p \in [1,\infty)$ and let $\varphi : L^p \to \mathbf{R}$ be a linear continuous function. Then there exist a unique $u \in L^{p'}$ such that

$$\varphi(f) = \int uf \qquad \forall f \in L^p$$

Thus, the topological dual of L^p is $L^{p'}$.

2.8. **Definition.** $L^p_{\text{loc}}(\Omega) :=$ functions such that $f\chi_K \in L^p(\Omega)$ for all compacts $K \subseteq \Omega$

2.9. **Fundamental lemma.** Let $f \in L^1_{\text{loc}}(\Omega)$ such that

$$\int fu = 0 \qquad \forall u \in C_c(\Omega)$$

then $f = 0$ almost everywhere on Ω.

2.9. **Theorem (density).** The space $C_c(\Omega)$ is dense in $L^p(\Omega)$ for $1 \le p < \infty$.

2.10. **Theorem (convolution in L^p).** The *convolution* of $f, g : \mathbf{R}^N \to \mathbf{R}$ is defined by

$$(f*g)(x) := \int_{\mathbf{R}} f(x-y)g(y)\mathrm{d}y$$

It is well-defined in the following cases:

f	g	$f*g$
L^1	L^1	L^1
L^1	L^2	L^2
L^p	$L^{p'}$	$L^\infty \cap \mathscr{C}$

3. Hilbert spaces

3.1. **Definition.** A *real Hilbert space* is a real vector space H with an inner product (f,g), such that H is complete under the norm $\|f\| := \sqrt{(f,f)}$. Examples: $\mathbf{R}^N$ and $L^2(\Omega)$.

3.2. **Cauchy-Schwarz inequality.** $|(f,g)| \leq \|f\| \, \|g\|$

3.3. **Pythagoras theorem.** If $(f,g) = 0$ then $\|f+g\|^2 = \|f\|^2 + \|g\|^2$.
If the sequence f_k is pairwise orthogonal then $\|\sum_k f_k\|^2 = \sum_k \|f_k\|^2$.

3.4. **Parallelogram identity.** $\|f\|^2 + \|g\|^2 = \frac{1}{2}\left(\|f+g\|^2 + \|f-g\|^2\right)$

3.5. **Polarization identity.** $(f,g) = \frac{1}{4}\left(\|f+g\|^2 - \|f-g\|^2\right)$

3.6. **Theorem (projection on a closed convex set).** Let $C \subseteq H$ be a nonempty closed convex set. Then, for any $f \in H$ there exist a unique $g \in C$ that minimizes the distance to f. The point g is characterized by the relation $\forall h \in C, (f-g, h-g) \leq 0$.

3.7. **Definition.** Let $A \subseteq H$. The *orthogonal* of A is the set

$$A^\perp := \{f, \ \forall a \in A, (f,a) = 0\}$$

It is a closed subspace of H.

3.8. **Theorem (orthogonal decomposition).** Let $F \subseteq H$ be a closed subspace. Then any element of H decomposes in a unique way in the form $f = g + h$ where $g \in F$ and $h \in F^\perp$. Moreover, g is the projection of f on F and h is the projection of f on $F^\perp$.

3.9. **Theorem (Riesz).** For any $f \in H$, the map $u \mapsto (u,f)$ is a continuous linear map from H to $\mathbf{R}$. Conversely, if $\varphi : H \to \mathbf{R}$ is a continuous linear map, there exists a unique $f \in H$ such that $\varphi(u) = (u,f) \ \ \forall u \in H$.

Thus, the dual of H is isomorphic canonically to H.

3.10. **Definition.** A *Hilbert basis* of H is an orthonormal sequence of vectors e_n, $n \in \mathbf{N}$ which is total (it spans the whole space H).

3.11. **Theorem.** Any (separable) Hilbert space admits a Hilbert basis.

3.12. **Theorem ("Fourier" series).** Any element of f can be written in a unique way using a Hilbert basis

$$f = \sum_n c_n e_n.$$

The coefficients are $c_n = (f, e_n)$ and they satisfy **Parseval's identity** $\|f\|^2 = \sum_n \|c_n\|^2$.

3.13. **Corollary.** All (separable) Hilbert spaces are isomorphic (to $\ell^2(\mathbf{N})$).

3.14. **Continuity of linear forms.** Let $\lambda : H \to \mathbf{R}$ be a linear form. We say that is continuous if and only if there exists a constant C such that $|\lambda(f)| \leq C\|f\|$.

3.15. **Continuity of bilinear forms.** Let $a : H \times H \to \mathbf{R}$ be a bilinear form. We say that is continuous if and only if there exists a constant C such that $|a(f,g)| \leq C\|f\|\|g\|$. Moreover, if there exists a constant $c > 0$ such that $a(f,f) \geq c\|f\|^2$ then the form is said to be *coercive*.

3.16. **Theorem (Lax-Milgram, symmetric case).** Let E be an energy of the form

$$E(f) = \frac{1}{2} a(f,f) - b(f)$$

where a is a continuous bilinear, symmetric and coercive form, and b is a continuous linear form. Then there is a unique f that minimizes $E(f)$. Moreover, it is characterized by the relation

$$a(f,u) = b(u) \qquad \forall u \in H.$$

4. Distributions

4.1. Definitions. Let $\Omega \subseteq \mathbf{R}$ be a connected subset. We introduce the following spaces

$$\mathscr{D}(\Omega) := \mathscr{C}_c^\infty(\Omega)$$

$$\mathscr{D}'(\Omega) := \text{dual of } \mathscr{D}(\Omega)$$

The dual is taken with respect to a topology of $\mathscr{D}$ defined elsewhere. The elements of $\mathscr{D}'$ are called *distributions*.

4.2. Notation. If $T : \mathscr{D} \to \mathbf{R}$ is a distribution, we use the following notations for $T(\varphi)$:

$$T(\varphi) \quad = \quad (T,\varphi) \quad = \quad \int_\Omega T\varphi \quad = \quad \int_\Omega T(x)\varphi(x)\mathrm{d}x$$

4.3. Definition. We say that a sequence of distributions T_n converges to T if for any function $\varphi \in \mathscr{D}$ we have $(T_n,\varphi) \to (T,\varphi)$.

4.4. Examples.

1) To any function $f \in L^1_{\mathrm{loc}}(\Omega)$ we associate a distribution T_f defined by

$$(T_f,\varphi) := \int_\Omega f(x)\varphi(x)\mathrm{d}x.$$

   This association is injective, and denoted simply by $T_f = f$.

2) Dirac delta: $(\delta,\varphi) := \varphi(0)$

3) Derivative of Dirac delta: $(\delta',\varphi) := -\varphi'(0)$

4) Dirac comb: $(Ш,\varphi) := \sum_{k\in\mathbf{Z}} \varphi(2\pi k)$.

5) Principal value of $1/x$:

$$(\mathrm{vp}(1/x),\varphi) := \lim_{\varepsilon\to 0}\int_\varepsilon^\infty \frac{\varphi(x)-\varphi(-x)}{x}\mathrm{d}x$$

   or equivalently, $\mathrm{vp}(1/\mathrm{x}) := \ln(|x|)'$ (defined below)

4.5. Derivative of a distribution. $(T',\varphi) := (T,-\varphi')$

4.6. Proposition. This definition of the derivative coincides with the classical one: when $T = T_f$ for a differentiable f, then $T_f' = T_{f'}$.

4.7. Product of a function and a distribution. $(fT,\varphi) := (T,f\varphi)$
Observation: the product of two distributions cannot be defined in general.

5. Sobolev spaces

5.1. Definition. We define the following spaces, for $1 \le p \le \infty$ and $k = 1,2,3,\dots$

$$W^{k,p}(\Omega) := \left\{ \begin{array}{l} \text{functions } f:\Omega\to\mathbf{R} \text{ such that} \\ \text{all their distributional derivatives} \\ \text{of order } 0,\dots,k \text{ belong to } L^p(\Omega) \end{array} \right\}$$

$$H^k(\Omega) := W^{k,2}(\Omega).$$

The space $W^{k,p}(\Omega)$ is endowed with the following norm (written here for the 1D case $\Omega \subseteq \mathbf{R}$):

$$\|f\|_{k,p} := \left(\|f\|_p^p + \|f'\|_p^p + \cdots \|f^{(k)}\|_p^p\right)^{1/p}$$

5.2. Properties. The spaces $W^{k,p}$ are complete normed vector spaces. The spaces H^k are Hilbert spaces (thus, their norms can be computed using a Hilbert basis and Parseval's identity).

6. Bibliography

6.1. H. Brézis : *Analyse Fonctionnelle*
(canonical reference for functional analysis)

6.2. L. C. Evans : *Partial Differential Equations*
(canonical reference for PDE)

6.3. C. Gasquet, P. Witomski : *Analyse de Fourier et applications*
(nice exposition of the theory of distributions)

7. Exercices

E1. Find two counterexamples to Beppo Levi (missing the condition of monotonicity). One example using compactly-supported functions and another one using bounded functions.

E2. Find two counterexamples to the dominated convergence theorem (missing the dominating function). One example using compactly-supported functions and another one using bounded functions.

***E3.** Can you find a function $f(x,y)$ that is integrable along each variable and $\int\left(\int f(x,y)\mathrm{d}x\right)\mathrm{d}y = 2$ but $\int\left(\int f(x,y)\mathrm{d}y\right)\mathrm{d}x = 3$.

E4. Find functions satisfying each of the following conditions (or prove that they do not exist)

(a) $f \in L^{10}(\mathbf{R}), f \notin L^{12}(\mathbf{R})$
(b) $f \notin L^{10}(\mathbf{R}), f \in L^{12}(\mathbf{R})$
(c) $f \in L^{10}(\mathbf{R}), f \notin L^{12}(\mathbf{R}), f \in L^{14}(\mathbf{R})$
(d) $f \notin L^{10}(\mathbf{R}), f \in L^{12}(\mathbf{R}), f \notin L^{14}(\mathbf{R})$

E5. Prove that the dual of L^1 is L^∞.

****E6.** Prove that the dual of L^∞ is not L^1.

E7. Find a Hilbert basis of $L^2([0,T])$ consisting of sine and cosine functions.

E8. Find a Hilbert basis of $L^2([0,T])$ consisting of only sine functions.

E9. Find a Hilbert basis of $L^2([0,T])$ consisting of only cosine functions.

***E10.** Find a Hilbert basis of $L^2(\mathbf{R})$.

E11. The polarization identity allows to compute the scalar product from the associated norm. Does it imply that any normed space has an associated scalar product?

E12. Prove proposition 4.6 above.

E13. Define the shift of a distribution on $\mathbf{R}$ by a length a. (Analogous to the shift $f(x-a)$ of a function f).

E14. If $\tau_a T$ denotes the shift of T, prove that $T' = \lim_{h\to 0} \frac{T-\tau_h T}{h}$

E15. Write down the scalar product that leads to the Sobolev norm on section 6.

E16. Let $\Omega \subseteq \mathbf{R}^2$ and $f \in L^2(\Omega)$. Prove that there is a unique function u minimizing the energy

$$E(u) = \frac{1}{2}\int \|\nabla u(x,y)\|^2 \mathrm{d}x\mathrm{d}y - \int f(x,y)u(x,y)\mathrm{d}x\mathrm{d}y$$

and satisfying the condition $u = 0$ on $\partial\Omega$. Moreover, the optimal function satisfies Poisson equation $-\Delta u = f$. *(Hint: define an appropriate Hilbert space, and apply Lax-Milgram theorem.)*